Kuiper's theorem and operator algebras

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Outline I

1. Classical Kuiper's theorem

2. Kuiper's theorem and Hilbert C*-modules
   - Relation to K-theory

3. Manuilov algebras on Hilbert C*-modules

4. Roe algebras and Kuiper property for spaces
   - Uniform Roe algebras
   - Spaces with Kuiper property
   - Non-Kuiper spaces

First ideas

Kuiper's theorem says:

Theorem

For a separable Hilbert space H, the group of invertible operators $GL(\mathbb{B}(H)) \subset \mathbb{B}(H)$ is contractible in the operator norm.

We will give an idea of its proof, slightly distinct from the original, but more adopted for generalizations.

First, $GL(\mathbb{B}(H))$ is an open set in a Banach space, hence it has the homotopy type of a CW-complex. Thus, it is contractible $\Leftrightarrow$ any $f : S \to GL(\mathbb{B}(H))$ can be deformed to $S \to \mathbf{1} \in GL(\mathbb{B}(H))$, where S is a finite polyhedron (sphere). Since $GL(\mathbb{B}(H))$ is open, we can also deform the image $f(S)$ to a finite polyhedron with arbitrary small simplices. Suppose, $F^1, \dots, F^s$ are its vertices. In fact we will work with them and verify that the homotopy is extendable to the entire set $f(S)$.

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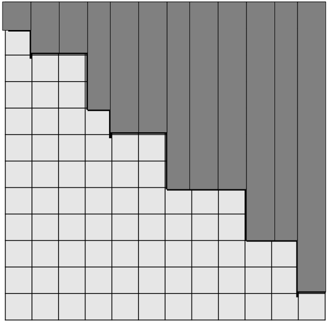
Second step

Fix an orthonormal base e_i in H (i.e., an identification $H \cong \ell_2$). Let $\|F^k_{ij}\|$ be the matrix of F^k in this base: $F^k e_i = \sum_j F^k_{ij} e_j$. An arbitrary small perturbation of columns $F^k e_i \in \ell_2$ (simultaneously for all F^k) allows us to suppose that the columns have finitely many non-zero entries, as at the picture:



Using the property, that elements of any row tend to zero, we can find inductively $i_1, j_1, i_2, j_2, \dots$ as at the next figure (left). Here the light-blue vectors are of length less $a, a/2, \dots$, for a sufficiently small a.

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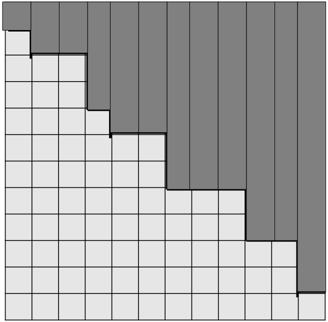
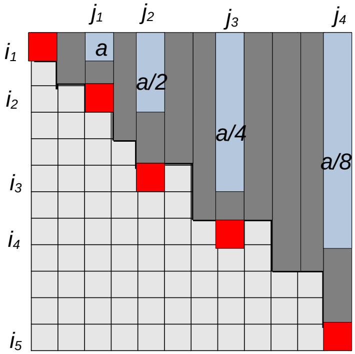


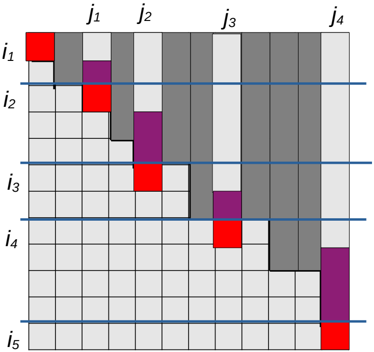
Using the property, that elements of any row tend to zero, we can find inductively $i_1, j_1, i_2, j_2, \dots$ as at the next figure (left). Here the light-blue vectors are of length less a, $a/2$, ..., for a sufficiently small a.

Third step



Then we can consider a linear homotopy of light-blue part to zero and obtain the following picture:

Fourth step



The new $F(e_{j_1})$, $F(e_{j_2})$, ... are in purple. We can rotate them to e_{i_1}, e_{i_2}, ... respectively, in orthogonal subspaces, which are "separated" by blue lines. And after that rotate e_{i_k} to e_{j_k} and arrive to a family (we denote the new operators still by F at each stage) of the following form.

Fifth step

For H_J generated by e_{j_k} and the orthogonal decomposition $H = (H_J)^\perp \oplus H_J$ (both summands are isomorphic to ℓ_2), we have

$$F = \begin{pmatrix} F' & 0 \\ F'' & 1 \end{pmatrix}, \qquad \text{and } \begin{pmatrix} F' & 0 \\ t\dot{F}'' & 1 \end{pmatrix} \text{ gives a path to } \begin{pmatrix} F' & 0 \\ 0 & 1 \end{pmatrix}$$

Decompose $H_J = H_0 \oplus H_1 \oplus \cdots$ into an orthogonal sum with each $H_i \cong \ell_2$. So, $F = diag(F', 1, 1, 1, \dots)$. Connect $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} F'(F')^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ at each summand $H_{2i} \oplus H_{2i+1}$ with $\begin{pmatrix} (F')^{-1} & 0 \\ 0 & F' \end{pmatrix}$ via the homotopy, for $t \in [0, \pi/2]$,

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} F' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} (F')^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

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Sixth step

The resulting operator is $\mathit{diag}(F', (F')^{-1}, F', (F')^{-1}, \dots)$. Applying the above homotopy in the inverse direction for sums $H_J \oplus H_0$, $H_1 \oplus H_2$, ...we arrive to $\mathbf{1} \in GL(\mathbb{B}(H))$ and the theorem is proved.

Hilbert C*-modules vs Hilbert spaces

Definition

A Hilbert C^*-module is a (right) Banach $\mathcal{A}$-module M over a C^*-algebra $\mathcal{A}$, equipped with an $\mathcal{A}$-valued inner product $M \times M \to \mathcal{A}$, $(m, n) \mapsto \langle m, n \rangle$ (with natural properties like $\langle m, n \rangle = \langle n, m \rangle^*$) such that the norm is given by $\|m\| = \|\langle m, m \rangle\|^{1/2}$.

We will be interested in the module $H_\mathcal{A}$, or $\ell_2(\mathcal{A})$, formed by all sequences $(a_1, a_2, \dots)$, $a_i \in \mathcal{A}$, such that $\sum_i (a_i)^* a_i$ is norm-convergent in $\mathcal{A}$ and

$$\langle (a_1, a_2, \dots), (b_1, b_2, \dots) \rangle = \sum_i (a_i)^* b_i.$$

There are the following main differences of Hilbert C^*-modules and Hilbert spaces:

1. Not each bounded homomorphism admits an adjoint.
2. Not each closed submodule has an orhogonal complement.

Hilbert C*-modules vs Hilbert spaces (continuation)

Example

Let $\mathcal{A} = C[0,1]$ be a Hilbert C^*-module over itself with the inner product a^*b. Then $C_0(0,1]$ is a proper closed submodule. But its orthogonal complement is trivial.

Example

For the same $\mathcal{A}$, one can consider an $\mathcal{A}$-functional, i.e. a morphism from $H_\mathcal{A}$ to $\mathcal{A}$, defined as $(a_1, a_2, \dots) \mapsto \sum_i f_i a_i$, where f_i are positive functions of norm 1 with non-intersecting supports. It is easy to see, that this is a bounded morphism (of norm 1) without an adjoint operator.

So, we have for $H_\mathcal{A}$ two algebras: $\mathbb{B}(H_\mathcal{A})$ and $\mathbb{B}^\star(H_\mathcal{A})$. Fortunately (at least for unital $\mathcal{A}$) we see, that a line of matrix becomes a column of the adjoint (plus conjugation), hence an element of $H_\mathcal{A}$. So the norm of elements in a line tends to zero. This is sufficient to apply the above approach in this situation and obtain a simple proof of:

Kuiper's theorem in Hilbert C*-modules

Theorem (Cuntz-Higson)

Suppose, $\mathcal{A}$ is σ-unital. Then $GL(\mathbb{B}^\star(H_\mathcal{A}))$ is contractible.

For a unital $\mathcal{A}$, this was obtained simultaneously by several authors, including myself.

For $GL(\mathbb{B}(H_\mathcal{A}))$ there are only partial results. In particular, we have proved the contractibility for $\mathcal{A} = C(X)$, where X is a finite-dimensional space and for $\mathcal{A} = \mathcal{K} \oplus \mathbb{C}$, where $\mathcal{K}$ is the algebra of compact operators,

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Outline

Relation to K-theory

The Kuiper theorems have the main application in K-theory and index theory as the key ingredient of the proof that the space of Fredholm operators $\mathcal{F}$ (respectively, the space of Mishchenko-Fomenko Fredholm operators $\mathcal{F}_{\mathcal{A}}$) is the classifying space for $K(X)$ (respectively, for $K(X;\mathcal{A})$)):

$$K(X) \cong [X, \mathcal{F}], \qquad K(X;\mathcal{A}) \cong [X, \mathcal{F}_{\mathcal{A}}].$$

In particular, the index of Fredholm operators

$$\mathrm{index} : \mathcal{F} \to \mathbb{Z}, \qquad \mathrm{index}_{\mathcal{A}} : \mathcal{F}_{\mathcal{A}} \to K(\mathcal{A}),$$

is a bijection on the connected components of the set of Fredholm operators.

Manuilov algebra

In [V. Manuilov, 2019, JMAA] the following C^*-algebra was introduced. Let H be a separable Hilbert space with a fixed orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. For $k\in\mathbb{N}$, denote by $\mathbb{B}_L^{(k)}(H)$ (resp. $\mathbb{B}_C^{(k)}(H)$) the set of all bounded operators on H such that each line (resp. each column) of their matrix (with respect to the fixed basis) contains no more than k non-zero entries. Note that

$$\mathbb{B}_L^{(k)}(H)\subset\mathbb{B}_L^{(l)}(H) \text{ when } k<l, \text{ and } \mathbb{B}_C^{(k)}(H)=(\mathbb{B}_L^{(k)}(H))^*.$$

Set also

$$\mathbb{B}^{(k)}(H)=\mathbb{B}_L^{(k)}(H)\cap\mathbb{B}_C^{(k)}(H).$$

Let $a,b\in\mathbb{B}(H)$, $A=(a_{ij})$, $B=(b_{ij})$ their matrices. Then evidently, if $a,b\in\mathbb{B}_L^{(k)}(H)$ then $a+b\in\mathbb{B}_L^{(2k)}(H)$. A more interesting property from V. Manuilov's paper is:

Manuilov algebra (continuation)

Lemma

If $a, b \in \mathbb{B}_L^{(k)}(H)$ then $ab \in \mathbb{B}_L^{(k^2)}(H)$.

Proof.

Let $c_{il} = \sum_{j\in\mathbb{N}} a_{ij} b_{jl}$. Fix i. There exist $j_1, \dots, j_k \in \mathbb{N}$ such that $a_{ij} = 0$ if $j \notin \{j_1, \dots, j_k\}$. For each j_m, $m = 1, \dots, k$, there exist $l_1^{(m)}, \dots, l_k^{(m)} \in \mathbb{N}$ such that $b_{j_m l} = 0$ if $l \notin \{l_1^{(m)}, \dots, l_k^{(m)}\}$. So $c_{il} = 0$ for $l \notin \{l_n^{(m)}\}_{n,m=1}^{k}$, hence the i-th line contains no more than k^2 non-zero entries. □

Let $\mathbb{B}_f(H)$ is the norm-closure of $\cup_k \mathbb{B}^{(k)}(H)$. This is a C^*-algebra.
V. Manuilov has proved (among the other statements) that the group of invertibles is contractible.

Generalizations to Hilbert C*-modules

For the Hilbert C^*-module $H_\mathcal{A}$ (it is natural to consider a unital $\mathcal{A}$) we can consider several generalizations: strong ones and weak ones.

Definition

Denote by $\mathbb{B}_L^{(k)}(H_\mathcal{A})$ the set of operators in $\mathbb{B}(H_\mathcal{A})$ having no more than k non-zero elements in each line of their matrices, and by $\mathbb{B}_C^{(k)}(H_\mathcal{A})$ the set of operators in $\mathbb{B}(H_\mathcal{A})$ having no more than k non-zero elements in each column of their matrices. Put $\mathbb{B}^{(k)}(H_\mathcal{A}) = \mathbb{B}_L^{(k)}(H_\mathcal{A}) \cap \mathbb{B}_C^{(k)}(H_\mathcal{A})$ Denote

$$\mathbb{B}_L^{\infty}(H_\mathcal{A}) := \bigcup_k \mathbb{B}_L^{(k)}(H_\mathcal{A}), \quad \mathbb{B}_C^{\infty}(H_\mathcal{A}) := \bigcup_k \mathbb{B}_C^{(k)}(H_\mathcal{A}),$$

$$\mathbb{B}^{\infty}(H_\mathcal{A}) := \bigcup_k \mathbb{B}^{(k)}(H_\mathcal{A}).$$

Generalizations to Hilbert C*-modules (continuation)

Definition

For a positive functional $\varphi : \mathcal{A} \to \mathbb{C}$, denote by φ_a the positive functional $\varphi_a(b) = \varphi(aba^*)$, where $a \in \mathcal{A}$..

Definition

Denote by $W\mathbb{B}_L^{(k)}(H_\mathcal{A}) \subset \mathbb{B}(H_\mathcal{A})$ (weakly having no more than k non-zero elements in line) the set of all operators from $\mathbb{B}(H_\mathcal{A})$ such that for any pure state φ on $\mathcal{A}$ and any $d \in \mathcal{A}$ there is no more than k elements in any line of the matrix of the operator, say $a_i^{j_1}, \dots a_i^{j_k}$ in the i^{th} line, with the property $\varphi_d(a_i^{j_s}(a_i^{j_s})^*) \neq 0$. Similarly, define $W\mathbb{B}_C^{(k)}(H_\mathcal{A}) \subset \mathbb{B}(H_\mathcal{A})$ to bethe set of all operators from $\mathbb{B}(H_\mathcal{A})$ such that for any pure state φ on $\mathcal{A}$ and any element $d \in \mathcal{A}$, there is no more than k elements in any column of the matrix of the operator, say $a_{j_1}^i, \dots a_{j_k}^i$ in the i^{th} column, with the property $\varphi_d((a_i^{j_s})^* a_i^{j_s}) \neq 0$.

Properties of the generalizations

Definition

Denote $W\mathbb{B}^{(k)}(H_{\mathcal{A}}) := W\mathbb{B}_L^{(k)}(H_{\mathcal{A}}) \cap W\mathbb{B}_C^{(k)}(H_{\mathcal{A}})$ and
$W\mathbb{B}_L^{\infty}(H_{\mathcal{A}}) := \bigcup_k W\mathbb{B}_L^{(k)}(H_{\mathcal{A}})$, $\quad W\mathbb{B}_C^{\infty}(H_{\mathcal{A}}) := \bigcup_k W\mathbb{B}_C^{(k)}(H_{\mathcal{A}})$,
$W\mathbb{B}^{\infty}(H_{\mathcal{A}}) := \bigcup_k W\mathbb{B}^{(k)}(H_{\mathcal{A}})$.

Denote by $\mathbb{B}_L^f(H_{\mathcal{A}})$, $\mathbb{B}_C^f(H_{\mathcal{A}})$, $\mathbb{B}^f(H_{\mathcal{A}})$, $W\mathbb{B}_L^f(H_{\mathcal{A}})$, $W\mathbb{B}_C^f(H_{\mathcal{A}})$, and $W\mathbb{B}^f(H_{\mathcal{A}})$ the corresponding closures.

Theorem

The algebras $\mathbb{B}_L^f(H_{\mathcal{A}})$ and $\mathbb{B}^f(H_{\mathcal{A}})$ consist of adjointable operators, i.e. are subalgebras of the C^*-algebra $\mathbb{B}^\star(H_{\mathcal{A}})$. Moreover, $\mathbb{B}^f(H_{\mathcal{A}})$ is an involutive subalgebra, hence a C^*-algebra.
The algebras $\mathbb{B}_C^f(H_{\mathcal{A}})$, $W\mathbb{B}_L^f(H_{\mathcal{A}})$, $W\mathbb{B}_C^f(H_{\mathcal{A}})$, and $W\mathbb{B}^f(H_{\mathcal{A}})$ generally contain non-adjointable operators.

Contractibility

Theorem

The following groups are contractible:

- $GL(\mathbb{B}^f_C(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A}))$
- $GL(\mathbb{B}^f_L(H_\mathcal{A}))$
- $GL(\mathbb{B}^f(H_\mathcal{A}))$
- $GL(W\mathbb{B}^f_C(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A}))$
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- $GL(W\mathbb{B}^f(H_\mathcal{A})) \cap \mathbb{B}^\star(H_\mathcal{A})$

Theorem

The group $GL(W\mathbb{B}^f(H_\mathcal{A}))$ is contractible inside $GL(\mathbb{B}(H_\mathcal{A}))$.

Outline

Roe algebras

Let (X, d) be a (countable) discrete metric space. Then the unit functions supported at one point δ_x, $x \in X$, form the standard base of the corresponding ℓ^2 space $\ell^2(X)$. For a bounded operator $F : \ell^2(X) \to \ell^2(X)$, let $(F_{xy})_{x,y \in X}$ denote the matrix of F with respect to the base $\{\delta_x\}_{x \in X}$.

Definition

Denote by $\mathcal{P}(F)$ the propagation of F, i.e. $\mathcal{P}(F) = \sup\{d(x,z) : x, z \in X, F_{xz} \neq 0\}$.

Note that the triangle inequality $d(x,y) \leq d(x,z) + d(z,y)$ implies $\mathcal{P}(FG) \leq \mathcal{P}(F) + \mathcal{P}(G)$.

Definition

The C^*-algebra $C^*_u(X)$ generated by operators of finite propagation in the algebra $\mathbb{B}(\ell_2(X))$ of all bounded operators is called the uniform Roe algebra.

Roe algebras

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Definition

The C^*-algebra $C^*_u(X)$ generated by operators of finite propagation in the algebra $\mathbb{B}(\ell_2(X))$ of all bounded operators is called the uniform Roe algebra.

Kuiper property: first statements

Definition

If $U(C^*_u(X))$ (equivalently, the group of invertibles of $C^*_u(X)$) is contractible, we say that (X, d) is a Kuiper space.

Proposition

Suppose (X, d) is a finite metric space. Then the group of invertibles in $C^*_u(X)$ is not contractible.

Proof.

In this case $C^*_u(X)$ is the matrix algebra $M_n(\mathbb{C})$, where $n = |X|$, and its invertibles form the group $GL_n(\mathbb{C})$, which is homotopy equivalent to the unitary group $U_n(\mathbb{C})$. Its fundamental group is not trivial (in fact $\cong \mathbb{Z}$) due to the epimorphism $\det : U_n(\mathbb{C}) \to S^1 \subset \mathbb{C}$. □

Kuiper property: first statements (continuation)

Proposition

Suppose, (X, d) is an infinite metric space of finite diameter. Then the group of invertibles in $C^*_u(X)$ is contractible.

Proof.

In this case $C^*_u(X) = \mathbb{B}(\ell_2(X))$ and the statement is exactly the original Kuiper theorem. □

Proposition

Suppose, $f : (X, d) \to (Y, \rho)$ is a bijection that is a coarse equivalence of metrics (i.e. there exist functions ϕ_1 and ϕ_2 on $[0, \infty)$ with $\lim_{t\to\infty} \phi_i(t) = \infty$, $i = 1, 2$, such that $\phi_1(d(x, y)) \le \rho(f(x), f(y)) \le \phi_2(d(x, y))$ for any $x, y \in X$). Then $C^*_u(X) \cong C^*_u(Y)$, in particular, Y is a Kuiper space if and only if so is X.

Kuiper property: first statements (continuation)

Proposition

Suppose, (X, d) is an infinite metric space of finite diameter. Then the group of invertibles in $C_u^*(X)$ is contractible.

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Kuiper property: one more statement

Definition

We say that a subset Y of (X, d) is r-sparse, if, for any $y \in Y$, $B_r(y) = \{y\}$.

Theorem

Suppose, for any r, there exists a subspace X_r of (X, d) such that

1) *X_r is a Kuiper space;*
2) *$X \setminus X_r$ is r-sparse.*

Then X is a Kuiper space.

Outline

Two properties

One can prove that the following two properties are equivalent:

Definition

We say that a discrete metric space (X, d) is PIUBS (has a countable partition by infinite uniformly bounded sets) if there exists a sequence of its points $\{x(k)\}_{k\in\mathbb{N}}$, a finite number $r > 0$, and a collection of sets $D_k \subset X$ such that

1) $\{D_k\}_{k\in\mathbb{N}}$ is a partition of X;
2) $x(k) \in D_k \subseteq B_r(x(k))$ for each k (in particular, $\{x(k)\}_{k\in\mathbb{N}}$ is a countable r-net for X), where $B_r(y)$ denotes the closed ball of radius r centered at y;
3) each D_k contains infinitely many points.

and

PIUBS and CIUBB spaces are Kuiper spaces

Definition

We say that a discrete metric space (X, d) is CIUBB (has a cover by infinite uniformly bounded balls) if there exists a sequence of its points $\{x(k)\}_{k\in\mathbb{N}}$ and a finite number $r > 0$ such that

1. The balls $B_r(x(k))$, $k \in \mathbb{N}$, form a cover of X (i.e. $\{x(k)\}$ is an r-net for X).
2. Each ball $B_r(x(k))$, $k \in \mathbb{N}$, contains infinitely many points.

Theorem

If X is PIUBS (or, equivalently CIUBB) then the group of invertibles in $C^*_u(X)$ is contractible.

Outline

Non-Kuiper spaces: preliminaries

For a metric space X, let $\sqcup^n X$ denote the space $X_1 \sqcup \ldots \sqcup X_n$, where $\alpha_i : X_i \to X$, $i = 1, \ldots, n$, are isometries, with the metric given by $d(x, y) = d_X(\alpha_i(x), \alpha_j(y)) + |i - j|$, where $x \in X_i$, $y \in X_j$. Then

$$C^*_u(\sqcup^n X) \cong M_n(C^*_u(X)).$$

Definition

We call X stable if for any $n \in \mathbb{N}$ there exists a bijection $\beta_n : \sqcup^n X \to X$ which is a coarse equivalence of metrics. For stable X, β_n induces an isomorphism $M_n(C^*_u(X)) \cong C^*_u(X)$ for any $n \in \mathbb{N}$ (by the above Proposition). X is locally finite (or proper) if each ball contains a finite number of points. For a subset $Y \subset X$ set $\partial_R Y = \{x \in X : d(x, Y) < R; d(x, X \setminus Y) < R\}$. Recall that X satisfies the Fölner property if for any $R > 0$ and any $\varepsilon > 0$ there exists a finite subset $F \subset X$ such that $\frac{|\partial_R F|}{|F|} < \varepsilon$.

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Non-Kuiper spaces from the Fölner trace

If X is locally finite then, for $T \in C^*_u(X)$ and for a finite set $F \subset X$ put $f_F(T) = \frac{1}{|F|}\sum_{x\in F} T_{xx}$. For a sequence of finite sets $F_n \subset X$ and an ultrafilter ω on $\mathbb{N}$, one can define the ultralimit $\lim_\omega f_{F_n}(T)$. Then it is known that the Fölner property allows to define in this way a trace f on $C^*_u(X)$ with $f(1) = 1$.

Theorem

Let X be a stable, locally finite metric space with the Fölner property. Then X is not a Kuiper space.

Proof.

If the group $GL(C^*_u(X))$ of invertible elements is contractible then, by stability of X, so are $GL(M_n(C^*_u(X)))$ for any $n \in \mathbb{N}$. One has $K_0(A) = \pi_1(\text{inj}\lim_{n\to\infty} GL(M_n(A)))$ for any unital Banach algebra A, hence $K_0(C^*_u(X)) = 0$. But $f(1) \neq f(0)$, hence $[1] \neq [0]$ in $K_0(C^*_u(X))$. □

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Thank you!