Phase transitions in Quantum many-body System

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  - $-268.8 \leq T \leq -38.8$; Normal conductors.
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  - Universal properties of different models.

Mathematical models in statistical physics

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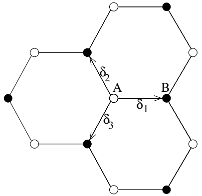
- We will introduce some mathematically rigorous results in the Hubbard model in the 2-D Honeycomb lattice.

The Hubbard model on the 2-d Honeycomb lattice.



- The honeycomb lattice $\Lambda = \Lambda^A \cup \Lambda^B$ is the superposition of the triangular lattice Λ^A (White dots) with $\Lambda^B = \Lambda^A + \vec{\delta}_i$ (Black dots): $\vec{\delta}_1 = (1, 0)$, $\vec{\delta}_2 = \frac{1}{2}(-1, \sqrt{3})$, $\vec{\delta}_3 = \frac{1}{2}(-1, -\sqrt{3})$.

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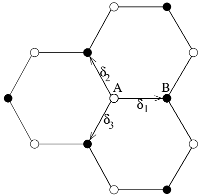

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- Let $L \in \mathbb{N}$, define the finite honeycomb lattice of side L: $\Lambda_L = \Lambda / L\Lambda$, $\lim_{L\to\infty} \Lambda_L = \Lambda$.

The space of states

- The one-particle Hilbert space $\mathcal{H}_L = \{\Psi_{\mathbf{x},\alpha,\tau} : \Lambda_L \times \{A, B\} \times \{\uparrow,\downarrow\} \to \mathbb{C}\ \}$ such that $\|\Psi\|_2^2 = \sum_{\mathbf{x},\tau,\alpha} |\Psi_{\mathbf{x},\alpha,\tau}|^2 = 1$, $\Lambda_L = \Lambda/L\Lambda$.

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- The Fermionic Fock space $\mathcal{F}_L$ is:

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$$(a^+_{\mathbf{z},\tau}\Psi)^{(N)}(\xi_1, \cdots, \xi_N) =$$

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} (-1)^j \delta_{\mathbf{z},\mathbf{x}_j} \delta_{\tau,\tau_j} \Psi^{(N-1)}(\xi_1, \cdots, \xi_{j-1}, \xi_{j+1}, \cdots, \xi_N),$$

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- The CAR: $\{a^+_{\mathbf{x},\tau}, a^-_{\mathbf{x}',\tau'}\} = \delta_{\mathbf{x},\mathbf{x}'}\delta_{\tau,\tau'}$, $\{a^+_{\mathbf{x},\tau}, a^+_{\mathbf{x}',\tau'}\} = 0$, $\{a^-_{\mathbf{x},\tau}, a^-_{\mathbf{x}',\tau'}\} = 0$. The same for $b^\pm_{\mathbf{z},\tau}$.

The Hubbard model on the honeycomb lattice

The Hubbard model Hamiltonian is:

$$\begin{aligned} H_{\Lambda_L} &= -t \sum_{\substack{\mathbf{x}\in\Lambda_A \\ i=1,2,3}} \sum_{\tau=\uparrow\downarrow} \left(a^+_{\mathbf{x},\tau} b^-_{\mathbf{x}+\vec{\delta}_i,\tau} + b^+_{\mathbf{x}+\vec{\delta}_i,\tau} a^-_{\mathbf{x},\tau} \right) \\ &- \mu \sum_{\mathbf{x}\in\Lambda_A} \sum_{\tau=\uparrow\downarrow} \left(a^+_{\mathbf{x},\tau} a^-_{\mathbf{x},\tau} + b^+_{\mathbf{x}+\vec{\delta}_i,\tau} b^-_{\mathbf{x}+\vec{\delta}_i,\tau} \right) \\ &+ \lambda \sum_{\mathbf{x}\in\Lambda_A} \left(a^+_{\mathbf{x},\uparrow} a^-_{\mathbf{x},\uparrow} a^+_{\mathbf{x},\downarrow} a^-_{\mathbf{x},\downarrow} + b^+_{\mathbf{x},\uparrow} b^-_{\mathbf{x},\uparrow} b^+_{\mathbf{x},\downarrow} b^-_{\mathbf{x},\downarrow} \right) \end{aligned}$$

- $t \in \mathbb{R}^+$ is called the hopping parameter, $\lambda \in \mathbb{R}$ is called the coupling constant, $\mu \in \mathbb{R}$ is called the chemical potential.

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- When $\lambda = 0$, any fermion is only hopping to its nearest neighbor. When $\lambda > 0$, all fermions are correlated through the interaction term.

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- It is easy to define yet highly nontrivial; It is mathematically challenging.
- This model captures the essence of many phenomena exhibited in the *Graphene* (Geim, Novoselov, 2004, Nobel Prize in Physics 2010), a mono-layer graphite, such as Dirac fermion, topological insulator, semi-metal, high-T_c superconductivity...

Time evolution and the correlation functions

- Define $\mathbf{a}_{x,\alpha}^{\pm}$, $\alpha = 1, 2$, s.t. $\mathbf{a}_{x,1}^{\pm} = \mathbf{a}_{x}^{\pm}$, $\mathbf{a}_{x,2}^{\pm} = \mathbf{b}_{x}^{\pm}$, the imaginary-time evolution of $\mathbf{a}_{x,\alpha}^{\pm}$ is: $\mathbf{a}_{x,\alpha}^{\pm} = e^{H_{\Lambda_L} x^0} \mathbf{a}_{\mathbf{x},\alpha}^{\pm} e^{-H_{\Lambda_L} x^0}$, $x = (x^0, \mathbf{x})$, $x_0 \in [-\beta, \beta)$, $\beta = 1/T$.

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- The Gibbs state associated with the Hamiltonian H_{Λ_L} is: $\langle \cdot \rangle = \mathrm{Tr}_{\mathcal{F}_L} [\, \cdot \, e^{-\beta H_{\Lambda_L}}]/Z_{\beta,\Lambda_L}$, $Z_{\beta,\Lambda_L} = \mathrm{Tr}_{\mathcal{F}_L} e^{-\beta H_{\Lambda_L}}$.

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- the n-point Schwinger function is defined as:

$$S_{n,\beta}(x_1, \alpha_1 \cdots x_n, \alpha_n, \lambda, \mu) = \lim_{L\to\infty} \langle \mathbf{T} \mathbf{a}^{\varepsilon_1}_{x_1,\alpha_1} \cdots \mathbf{a}^{\varepsilon_n}_{x_n,\alpha_n} \rangle_{\beta,L}$$

$$\mathbf{T} \mathbf{a}^{\varepsilon_1}_{(\mathbf{x}_1, x^0_1),\alpha_1} \cdots \mathbf{a}^{\varepsilon_n}_{(\mathbf{x}_n, x^0_n),\alpha_n,\tau_n} = \mathrm{sgn}(\pi)\ \mathbf{a}^{\varepsilon_{\pi(1)}}_{(\mathbf{x}_{\pi(1)}, x^0_{\pi(1)}),\alpha_{\pi(1)}} \cdots \mathbf{a}^{\varepsilon_{\pi(n)}}_{(\mathbf{x}_{\pi(n)}, x^0_{\pi(n)}),\alpha_{\pi(n)}},$$

is the time-ordering operator, $\mathrm{sgn}(\pi)$ is the sign of the permutation π, in which $x^0_{\pi(1)} \geq x^0_{\pi(2)} \geq \cdots \geq x^0_{\pi(n)}$.

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  - The regularity of $S_{2,\beta}(x_1,x_2,\lambda,\mu)$ as a function of λ, β and the coordinates x_1,x_2.

The Hubbard model at $\mu = 0$.

Theorem (Giuliani, Mastropietro, 2010)

There exists a positive constant U *such that the "pressure function"* $\log \frac{Z_{\beta,\Lambda}(\lambda)}{Z_{\beta,\Lambda}(0)}$ *and the connected Schwinger function* $S^c_{2,\beta}(x_1, x_2, \lambda)$ *are both analytic functions of* λ *when* $\beta \to \infty$*, for* $|\lambda| \leq U$*. The ground state is a fermi liquid up to* $T = 0$*.*

Definition (Fermi liquid, Salmhofer, 1998)

Let $\hat{S}^c_{2,\beta}(k, \lambda)$ be the Fourier transform of $S^c_{2,\beta}(x_1, x_2, \lambda)$. The ground state of an interacting many-fermion system is said to be a Fermi liquid if

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Definition (Fermi liquid, Salmhofer, 1998)

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- $\hat{S}_{2,\beta}(k, \lambda)$ is an analytic function of the coupling constant λ for $\beta < \infty$.
- The self-energy $\hat{\Sigma}(k, \lambda)$ function is analytic in λ and uniformly C^2 in k for $\beta \to \infty$.

The Honeycomb-Hubbard model at $\mu = 1$, $\lambda = 0$

- $$\hat{S}_{2,\beta}(k_0, \mathbf{k}, 0) = \frac{1}{k_0^2 + |\Omega(\mathbf{k})|^2 - \mu^2 - 2i\mu k_0} \begin{pmatrix} ik_0 + \mu & -\Omega^*(\mathbf{k}) \\ \Omega(\mathbf{k}) & ik_0 + \mu \end{pmatrix}$$

  $k_0 = (2n+1)\pi/\beta$, $\mathbf{k} = (k_1, k_2) \in \mathbb{R}^2/\Lambda$,
  $\Omega(\mathbf{k}) = 1 + 2e^{-i\frac{3}{2}k_1}\cos(\frac{\sqrt{3}}{2}k_2)$ is called the dispersion relation.

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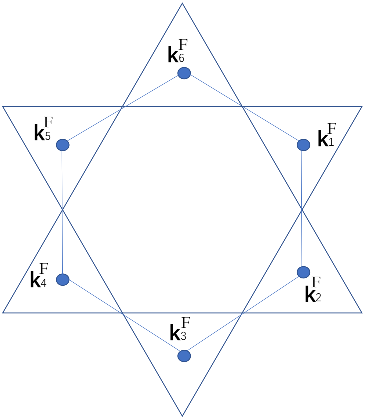
- It is well defined for any $\beta < \infty$.
- For $k_0 \to 0$ ($\beta \to \infty$), $\mu = 1$, $\hat{S}_{2,\beta}(k_0, \mathbf{k}, 0)$ is singular on the Fermi surface

$$\begin{aligned} \mathcal{F} &= \{\mathbf{k} \in \mathbb{R}^2/\Lambda, |\Omega(\mathbf{k})| - 1 = 0\} \\ &= \{(k_1, k_2), k_2 = \pm \frac{(2n+1)\pi}{\sqrt{3}}, n \in \mathbb{Z}\} \\ &\qquad \cup \{(k_1, k_2), k_2 = \pm\sqrt{3}k_1 \mp \frac{4n+2}{\sqrt{3}}\pi,\ n \in \mathbb{Z}\}. \end{aligned}$$

The Fermi surface



The Honeycomb Hubbard model at $\mu = 1$, $\lambda \neq 0$.

Theorem (Rivasseau, ZW 2021)

- *There exists a positive constants* $\beta_c = 1/T_c$ *such that for any* $\beta \leq \beta_c$, *the "pressure function"* $\log \frac{Z_{\beta,\Lambda}(\lambda)}{Z_{\beta,\Lambda}(0)}$ *and the connected two-point function* $S^c_{2,\beta}(\lambda)$ *are analytic functions of the coupling constant* λ, *in the region*

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- *The self-energy function $\hat{\Sigma}(k,\lambda)$ is $C^{1+\epsilon}$ differentiable w.r.t. the momentum, so that the ground state is not a Fermi liquid.*

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- $\Omega(\mathbf{k})$ and μ are deformed by interactions, $\mathcal{F}$ is not fixed.
- The perturbation series can be unbounded.

Proof of the main theorem -The Grassmann algebra

- The Grassmann algebra **Gra** is an associative, non-commutative, nilpotent algebra generated by the Grassmann variables $\{\hat{\psi}^{\varepsilon}_{k,\alpha}\}$, $k = (k_0, \mathbf{k})$, $\varepsilon = \pm$, $\alpha = 1, 2$ such that $\hat{\psi}^{\varepsilon}_{k,\alpha}\hat{\psi}^{\varepsilon'}_{k',\alpha'} = -\hat{\psi}^{\varepsilon'}_{k',\alpha'}\hat{\psi}^{\varepsilon}_{k,\alpha}$ and $(\hat{\psi}^{\varepsilon}_{k,\alpha})^2 = 0$.

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- $\int d\psi^+_\alpha d\psi^-_b e^{-\psi^+_\alpha C_{\alpha\beta}\psi^-_b}=C_{\alpha\beta}$ for $C_{\alpha\beta}\in\mathbb{C}$

The Grassmann algebra

The Grassmann Gaussian measure $P(d\psi)$ with covariance $\hat{C}(k) := \hat{S}_{2,\beta}(k,0)$ is defined by :

$$P(d\psi) = N^{-1} D\psi \cdot \exp\left\{ -\frac{1}{|\Lambda_L|\beta} \sum_{k\in\mathcal{D}_{\beta,L},\tau=\uparrow\downarrow,\alpha=1,2} \hat{\psi}^+_{k,\tau,\alpha} \hat{C}(k)^{-1} \hat{\psi}^-_{k,\tau,\alpha} \right\}$$

where

$$N = \prod_{\mathbf{k}\in\mathcal{D}_L,\tau=\uparrow\downarrow} \frac{1}{\beta|\Lambda_L|} \begin{pmatrix} -ik_0 - 1 & -\Omega^*(\mathbf{k}) \\ -\Omega(\mathbf{k}) & -ik_0 - 1 \end{pmatrix},$$

$\mathcal{D}_{\beta,L} = \mathcal{D}_\beta \times \mathcal{D}_L$, $\mathcal{D}_\beta = \{\frac{2\pi}{\beta}(n+\frac{1}{2}), n \in \mathbb{N}\}$, $\mathcal{D}_L$ is the dual space of Λ_L. We have:

$$\lim_{L\to\infty} \int P(d\psi) \hat{\psi}^-_{k_1,\tau_1,\alpha_1} \hat{\psi}^+_{k_2,\tau_2,\alpha_2} = \delta_{k_1,k_2} \delta_{\tau_1,\tau_2} [\hat{C}(k_1)]_{\alpha_1,\alpha_2}.$$

The Grassmann functional integrals

- Define the Grassmann fields
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- the interacting potential becomes:

$$\mathcal{V}(\psi) = \lambda \sum_{\alpha,\alpha'=1,2} \int_{\Lambda_{\beta,L}} d^3x \, \psi^{+}_{x,\uparrow,\alpha} \psi^{-}_{x,\uparrow,\alpha'} \psi^{+}_{x,\downarrow,\alpha} \psi^{-}_{x,\downarrow,\alpha'} \ ,$$

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- The Schwinger functions:

$$S_{n,\beta}(x_1,\cdots,x_n) = \frac{1}{Z}\int \psi^{\epsilon_1}_{x_1,\tau_1,\alpha_1}\cdots\psi^{\epsilon_n}_{x_n,\tau_n,\alpha_n} e^{-\lambda\mathcal{V}(\psi)}P(d\psi).$$

Generating functionals

- Let j^+, j^- be two Grassmann fields. Define $Z(j^+, j^-) = \int e^{-\lambda \mathcal{V}(\psi) + \int dx \psi^+(x) j^-(x) + + \int dx j^+(x) \psi^-(x)} P(d\psi)$, the generating functional of the Schwinger's functions.

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$$\begin{aligned} &S^c_{2n}(x_1, \cdots, x_n, y_1, \cdots, y_n) \\ &= \frac{\delta^{2n}}{\delta j^+(x_1)\cdots\delta j^+(x_n)\delta j^-(y_1)\cdots\delta j^-(y_n)} W(j^+, j^-)|_{j^\pm=0}, \end{aligned}$$

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- Define $\phi^+(x) = \frac{\delta}{\delta j^-(x)} W(j)$, $\phi^-(x) = \frac{\delta}{\delta j^+(x)} W(j)$. $\Gamma(\phi^+, \phi^-) = W(j^+, j^-) - \int d^3x \, [j^+(x)\phi^-(x) + \phi^+(x) j^-(x)]$ is defined as the Legendre transform of W.

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- The self-energy $\Sigma(x, y) = \frac{\delta^2}{\delta \phi^+(x) \delta \phi^-(y)} \Gamma(\phi^+, \phi^-)|_{\phi^{\pm}=0}$

The partition function

For $|\lambda| < 1$, we do perturbation expansions:

$$Z(\lambda) = S_0(\lambda) = \int P(d\psi) e^{\lambda \int_{\Lambda_{\beta,L}} d^3x \left[\psi^+_{x,\uparrow}\psi^-_{x,\uparrow}\psi^+_{x,\downarrow}\psi^-_{x,\downarrow}\right]}$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int P(d\psi) \Big[\int_{\Lambda_\beta} d^3x \,(\psi^+_{x,\uparrow}\psi^-_{x,\uparrow}\psi^+_{x,\downarrow}\psi^-_{x,\downarrow}) \Big]^n.$$

$$= \sum_n \frac{\lambda^n}{n!} \int_{(\Lambda_{\beta,L})^n} d^3x_1 \cdots d^3x_n \begin{Bmatrix} x_{1,\varepsilon_1,\tau_1} \cdots x_{n,\varepsilon_n,\tau_n} \\ x_{1,\varepsilon_1,\tau_1} \cdots x_{n,\varepsilon_n,\tau_n} \end{Bmatrix},$$

$\{\,\cdot\,\}$ is a $2n \times 2n$ determinant, Cayley's notation:

$$\begin{Bmatrix} x_{i,\tau} \\ x_{j,\tau'} \end{Bmatrix} = \det\left[\delta_{\tau\tau'}[\, C(x_i - x_j)]\right], C(x-y) = \int_{\Lambda_{\beta,L}} \hat{C}(k) e^{ik(x-y)} d^3x$$

$$\hat{C}(k) = \frac{1}{k_0^2 + |\Omega(\mathbf{k})|^2 - \mu^2 - 2i\mu k_0} \begin{pmatrix} ik_0 + \mu & -\Omega^*(\mathbf{k}) \\ \Omega(\mathbf{k}) & ik_0 + \mu \end{pmatrix}$$

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- Solution: partially expand the determinant (fermionic cluster expansions) so that only tree lines appear. Dividing the integral domain of $\hat{C}(k)$ into smaller regions (sectors), so that $\hat{C}(k)$ and its Fourier transform have optimal decaying property.

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  - Introducing counter-terms to fix the Fermi surface

$$\begin{aligned} \delta H_{\Lambda_L} &= \delta\mu(\lambda) \sum_{k\in\mathcal{D}_{\beta,L}} \sum_{\alpha=1,2} \sum_{\tau=\uparrow,\downarrow} \hat{\psi}^+_{k,\tau,\alpha}\hat{\psi}^-_{k,\tau,\alpha} \\ &+ \sum_{k\in\mathcal{D}_{\beta,L},\tau=\uparrow\downarrow} \sum_{\alpha,\alpha'=1,2} \hat{\nu}(k_0, \mathbf{k}, \lambda)\hat{\psi}^+_{k,\tau,\alpha}\hat{\psi}^-_{k,\tau,\alpha'} \end{aligned} \quad (2)$$

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- Solution:
  - Introducing counter-terms to fix the Fermi surface

$$\begin{aligned} \delta H_{\Lambda_L} &= \delta\mu(\lambda) \sum_{k \in \mathcal{D}_{\beta,L}} \sum_{\alpha=1,2} \sum_{\tau=\uparrow,\downarrow} \hat{\psi}^+_{k,\tau,\alpha} \hat{\psi}^-_{k,\tau,\alpha} \\ &+ \sum_{k \in \mathcal{D}_{\beta,L}, \tau=\uparrow\downarrow} \sum_{\alpha,\alpha'=1,2} \hat{\nu}(k_0, \mathbf{k}, \lambda) \hat{\psi}^+_{k,\tau,\alpha} \hat{\psi}^-_{k,\tau,\alpha'} \end{aligned} \quad (2)$$

  - Choose $\delta\mu(\lambda)$ such that it cancels the term $\tilde{\delta}\mu(\lambda)$.
  - Choose $\hat{\nu}(k_0, \mathbf{k}, \lambda)$ that cancels $\hat{\Sigma}(0, P_F(\mathbf{k}), \lambda)|$
  - The cancellations are carried in the multi-scale representation using renormalization theory.

The multi-scale analysis

- Let $G_0^h(\mathbb{R})$, $h > 1$, be the Gevrey class of compactly supported functions. Define a cutoff function $\chi \in G_0^h(\mathbb{R})$ as:

$$\chi(t) = \chi(-t) = \begin{cases} = 0\ , & \text{for} \quad |t| > 2, \\ \in (0,1)\ , & \text{for} \quad 1 < |t| \leq 2, \\ = 1, & \text{for} \quad |t| \leq 1. \end{cases} \tag{3}$$

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- Given fixed constant $\gamma \ge 10$, construct a partition of unity

$$\begin{aligned} 1 &= \sum_{j=0}^{\infty} \chi_j(t), \quad \forall t \ne 0; \qquad (4) \\ \chi_0(t) &= 1 - \chi(t), \\ \chi_j(t) &= \chi(\gamma^{2j-1} t) - \chi(\gamma^{2j} t) \text{ for } j \ge 1. \end{aligned}$$

The multi-slice expansion

- The free propagator is decomposed as :

$$\hat{C}(k)_{\alpha\alpha'} = \sum_{j=0}^{\infty} \hat{C}_j(k)_{\alpha\alpha'}, \ \alpha, \alpha' = 1, 2, \tag{5}$$

$$\hat{C}_j(k)_{\alpha\alpha'} = \hat{C}(k)_{\alpha\alpha'} \cdot \chi_j[4k_0^2 + e^2(\mathbf{k})],$$

$$e(\mathbf{k}) = 8[\cos(\sqrt{3}k_2/2)] \cdot [\cos(\frac{1}{4}(3k_1 + \sqrt{3}k_2))] \cdot [\cos(\frac{1}{4}(3k_1 - \sqrt{3}k_2))].$$

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- $\hat{C}_j(k) = \sum_{\sigma=(s_a,s_b)} \hat{C}_{j,\sigma}(k)$, $\hat{C}_{j,\sigma}(k) = \hat{C}_j(k) \cdot v_{s_a}[t_a] \; v_{s_b}[t_b]$,
  $a, b \in \{1,2,3\}$,
  $t_1 = \cos^2(\sqrt{3}k_2/2)$, $t_2 = \cos^2(\frac{1}{4}(3k_1 + \sqrt{3}k_2))$,
  $t_3 = \cos^2(\frac{1}{4}(3k_1 - \sqrt{3}k_2))$.

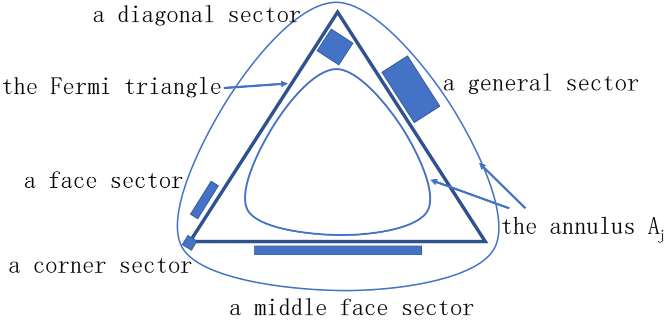

Figure: An illustration of the various sectors.

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  where $s_a, s_b \in [0,j]$, $\alpha_0 = 1/h$, and $d_{j,\sigma}(x,y) = \gamma^{-j}|x_0-y_0| + \gamma^{-s_a}|x_a-y_a| + \gamma^{-s_b}|x_b-y_b|$

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- $$\Big\|\ [C_{j,\sigma}(x)]_{\alpha\alpha'}\ \Big\|_{L^1} \leq O(1)\gamma^j.$$

The fermionic cluster expansions

- Recall that

$$Z(\lambda) = \int P(d\psi) e^{\lambda \int_{\Lambda_{\beta,L}} d^3x \left[\psi^+_{x,\uparrow}\psi^-_{x,\uparrow}\psi^+_{x,\downarrow}\psi^-_{x,\downarrow}\right]}$$

$$= \sum_n \frac{\lambda^n}{n!} \int_{(\Lambda_{\beta,L})^n} d^3x_1 \cdots d^3x_n \begin{Bmatrix} x_{1,\varepsilon_1,\tau_1} \cdots x_{n,\varepsilon_n,\tau_n} \\ x_{1,\varepsilon_1,\tau_1} \cdots x_{n,\varepsilon_n,\tau_n} \end{Bmatrix},$$

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- Partially expand the determinant such that only tree graphs appear.

Theorem (The BKAR jungle Formula. Brydges, Kennedy 87, Abdesselam Rivasseau 95)

Let $I_n = \{1, \cdots, n\}$, $\mathcal{P}_n = \{\ell = (i,j), i,j \in I_n, i \neq j\}$, $\mathcal{S}$ a set of smooth functions from $\mathbb{R}^{\mathcal{P}_n}$ to some Banach space, $\mathbf{1} \in \mathbb{R}^{\mathcal{P}_n}$ be the vector with every entry equals 1. Then for any $\mathbf{x} = (x_\ell)_{\ell \in \mathcal{P}_n} \in \mathbb{R}^{\mathcal{P}_n}$ and $f \in \mathcal{S}$:

$$f(\mathbf{1}) = \sum_{\mathcal{J}} \Big(\int_0^1 \prod_{\ell \in \mathcal{F}} dw_\ell \Big) \Big(\prod_{k=1}^{m} \Big(\prod_{\ell \in \mathcal{F}_k \setminus \mathcal{F}_{k-1}} \frac{\partial}{\partial x_\ell} \Big) \Big) f[X^{\mathcal{F}}(w_\ell)],$$

- $\mathcal{J} = (\mathcal{F}_0 \subset \mathcal{F}_1 \cdots \subset \mathcal{F}_{r_{max}} = \mathcal{F})$ *is any partially ordered set of forests* $\mathcal{F}_i$ *with* n *vertices.*
- $X^{\mathcal{F}}(w_\ell)$ *is a vector with elements* $x_\ell = x_{ij}^{\mathcal{F}}(w_\ell)$*:*
  - $x_{ij}^{\mathcal{F}} = 1$ *if* $i = j$*, or if* i *and* j *are connected by* $\mathcal{F}_{k-1}$*.*
  - $x_{ij}^{\mathcal{F}} = 0$ *if* i *and* j *are not connected by* $\mathcal{F}_k$*,*
  - $x_{ij}^{\mathcal{F}} = \inf_{\ell \in P_{ij}^{\mathcal{F}}} w_\ell$*, if* i *and* j *are connected by the forest* $\mathcal{F}_k$ *but not* $\mathcal{F}_{k-1}$*, where* $P_{ij}^{\mathcal{F}_k}$ *is the unique path in the forest that connects* i *and* j*,*

The connected functions

- $S^c_{2p} = \sum_n S^c_{2p,n} \lambda^n$,

$$S^c_{2p,n} = \frac{1}{n!} \sum_{\{\underline{\tau}\},\mathcal{G},\mathcal{T}} \sum_{\mathcal{J}}{}' \epsilon(\mathcal{J}) \prod_{j=1}^{n} \int d^3x_j \delta(x_1)$$
$$\prod_{\ell\in\mathcal{T}} \int_0^1 dw_\ell C_{\tau_\ell,\sigma_\ell}(x_\ell,\bar{x}_\ell) \prod_{i=1}^{n} \chi_i(\sigma) \,\mathrm{det}_{\mathrm{left}}(C_j(w)) \ .$$

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  - $\|S_{2,n}\|_{L^\infty} \le C_2 \log \frac{1}{T}$, *for* $n \ge 1$ *and* $\lambda < O(1)/|\log T|^2$.
- We need renormalizations for the $p = 1$ case.

The Multi-arch expansions for the self-energy

Theorem (Rivasseau, ZW 2021)

- *The amplitude of the self-energy is given by:*

$$\Sigma(y,z) = \sum_{n=0}^{\infty} \frac{\lambda^{n+2}}{n!} \int_{\Lambda^n} d^3x_1 \dots d^3x_n \sum_{\{\mathcal{I}\}} \sum_{\mathcal{G}_{\mathcal{B}}} \sum_{\mathcal{EB}} \sum_{\mathcal{T}} \sum_{\{\sigma\}}$$

$$\sum_{\substack{m-\text{arch systems}\\ \left((f_1,g_1),\dots,(f_m,g_m)\right)\\ \text{with } m\le p}} \left(\prod_{\ell\in\mathcal{T}} \int_0^1 dw_\ell\right) \left(\prod_{r=1}^{m} \int_0^1 ds_r\right) \left(\prod_{\ell\in\mathcal{T}} C_{\sigma(\ell)}(f_\ell, g_\ell)\right)$$

$$\left(\prod_{r=1}^{m} C(f_r,g_r)(s_1,\dots,s_{r-1})\right) \frac{\partial^m \det_{left,\mathcal{T}}}{\prod_{r=1}^m \partial C(f_r,g_r)} \left(\{w_\ell\},\{s_r\}\right) .$$

- $\exists C, K > 0$ *s.t.* $\|\Sigma(y,z)\|_{L^\infty} \le K \log \frac{1}{T}$, $\forall\ n \ge 1$, $\lambda < \frac{C}{|\log T|^2}$.

Need renormalization.

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- A systematic way of removing divergences in QMB and QFT.

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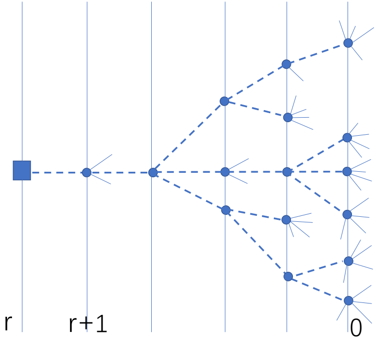

The renormalization of the two-point function

- Let ϕ_{e_1}, ϕ_{e_2} some smooth, compactly supported functions.

$$\begin{aligned}
&\int dy' dz\ \phi_{e_1}(y') S_r^c(y,z) \phi_{e_2}(z) \\
&= \int dy\ [\int dz\ S_r^c(y,z)] \phi_{e_1}(y) \phi_{e_2}(y) \\
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- Similarly, cancellation of the self-energy with the counter-term

$$\hat{\Sigma}^r_{s_+,s_-}\left[(2\pi T, P_F(\mathbf{k}))_{s_+,s_-}, \hat{\nu}^{\leq(r-1)}, \lambda\right] + \hat{\nu}^r_{s_+,s_-}(P_F(\mathbf{k})_{s_+,s_-}, \lambda) = 0.$$

The renormalizations are performed from the high scale terms to the low scale terms, following the Gallavotti-Nicolò tree.

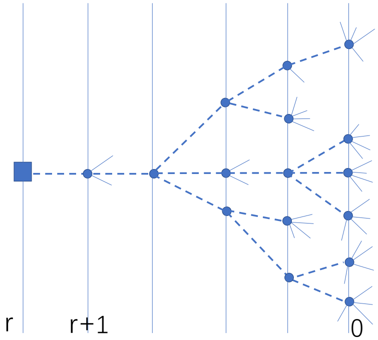

Figure: A Gallavotti-Nicolò tree with 16 nodes and 8 bare vertices. The round dots represent the nodes and bare vertices, and the big square represents the root, which has the scaling index $r \leq -1$. The dash lines are the inclusion relations between these nodes and the thin lines are the external fields of the nodes.

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- Theorem (ZW, 2022)

  For $\mu = 2 - \mu_0$, $\mu_0 \ll 1$, the ground state is a not a Fermi liquid for $T \geq T_c$, with $T_c = \frac{K_3}{\mu_0} \exp(-\frac{C_3}{|\lambda|^{1/2}})$.

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- Metal-Insulator transitions and many-body localization in Hubbard model.

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- Honeycomb Hubbard Model at van Hove Filling II: Lower Bounds of the Self-Energy, arXiv:2108.10415.

Thanks for your attention!