How to quantum shuffle cards
– mixing time and cutoff profiles

Simeng Wang

Harbin Institute of Technology

joint work with Amaury Freslon (Orsay) & Lucas Teyssier (Vienna)

1st Harbin-Moscow Conference on Analysis, July 2022

How to (classically) shuffle cards

Poincaré (1912): when we are playing cards, after a sufficiently long time, all the permutations of cards appear with equal probabilities.

CHAPITRE XVI.

QUESTIONS DIVERSES.

---

225. Battage des cartes. — Je me suis occupé dans l'introduction des problèmes relatifs au joueur qui bat un jeu de cartes. Pourquoi, quand le jeu a été battu assez longtemps, admettons-nous que toutes les permutations des cartes, c'est-à-dire tous les ordres dans lesquels ces cartes peuvent être rangées, doivent être également probables? C'est ce que nous allons examiner de plus près.

$\rightsquigarrow$ "random walk theory"

- Spread the cards on a table;
- Select one card uniformly at random, then put it back;
- Select a second card in the same way;
- Swap the two cards if different;
- Otherwise, do nothing.

Interpretation:

- μ_{tran} = uniform measure on transpositions in the permutation group S_N
- $\mu_N = \frac{N-1}{N}\mu_{\text{tran}} + \frac{1}{N}\delta_{\text{id}}$
- Random walk on S_N driven by μ_N: the distribution at the k-th step is

$$\mu_N^{*k}(\sigma) := \sum_{\substack{\sigma_1,\ldots,\sigma_k \in S_N \\ \sigma_1\cdots\sigma_k=\sigma}} \mu_N(\sigma_1)\cdots\mu_N(\sigma_k), \quad \sigma \in S_N.$$

A "LAZY" CARD SHUFFLE BY RANDOM TRANSPOSITIONS

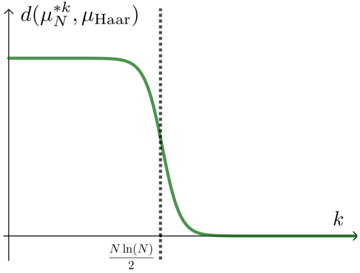
Question Does this acutally mix the cards ?
Answer Yes. μ_N^{*k} converges weakly to the Haar measure on S_N.

A "LAZY" CARD SHUFFLE BY RANDOM TRANSPOSITIONS

Question Does this acutally mix the cards ?
Answer Yes. μ_N^{*k} converges weakly to the Haar measure on S_N.

Question How and when?
Answer Diaconis-Shahshahani 81′:

- Before $N\ln(N)/2$ steps, the distribution stays far from uniform;
- After $N\ln(N)/2$ steps, the distribution suddenly drops to uniform.



"cutoff phenomenon"

Cutoff phenomenon for random transpositions

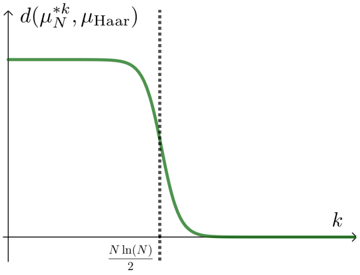
Denote μ_{Haar} = Haar measure on S_N. The total variation distance

$$d(\mu_N^{*k}, \mu_{\mathrm{Haar}}) := \sup_{A \subset S_N} |\mu_N^{*k}(A) - \mu_{\mathrm{Haar}}(A)| = \frac{1}{2}\|\mu_N^{*k} - \mu_{\mathrm{Haar}}\|_1$$

Theorem (Diaconis-Shahshahani 81′)

For $\epsilon > 0$, as $N \to \infty$

$$d(\mu_N^{*\lfloor(1-\epsilon)N\ln(N)/2\rfloor}, \mu_{\mathrm{Haar}}) \to 1, \quad d(\mu_N^{*\lfloor(1+\epsilon)N\ln(N)/2\rfloor}, \mu_{\mathrm{Haar}}) \to 0.$$



Cutoff phenomenon for random transpositions

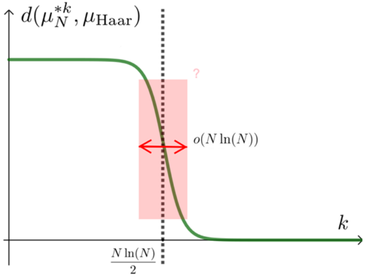
Denote μ_{Haar} = Haar measure on S_N. The total variation distance

$$d(\mu_N^{*k}, \mu_{\text{Haar}}) := \sup_{A \subset S_N} |\mu_N^{*k}(A) - \mu_{\text{Haar}}(A)| = \frac{1}{2}\|\mu_N^{*k} - \mu_{\text{Haar}}\|_1$$

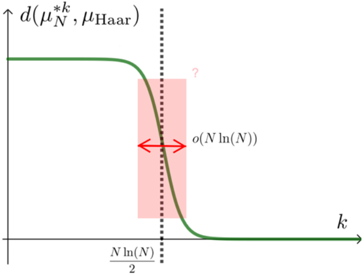
Theorem (Diaconis-Shahshahani 81′)

For $\epsilon > 0$, as $N \to \infty$

$$d(\mu_N^{*\lfloor(1-\epsilon)N\ln(N)/2\rfloor}, \mu_{\text{Haar}}) \to 1, \quad d(\mu_N^{*\lfloor(1+\epsilon)N\ln(N)/2\rfloor}, \mu_{\text{Haar}}) \to 0.$$

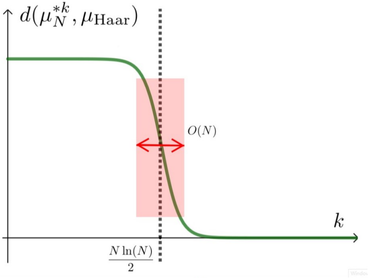


Cutoff profiles



How does the "cutoff" occur in the short window?

CUTOFF PROFILES



How does the “cutoff” occur in the short window?

Theorem (Teyssier, Ann. Proba. 20′)

For $c \in \mathbb{R}$ *and* $N \to \infty$,

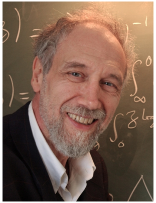
$$d(\mu_N^{*\frac{1}{2}(N\ln(N)+cN)}, \mu_{\text{Haar}}) \to d(\text{Poiss}(1+e^{-c}), \text{Poiss}(1)),$$

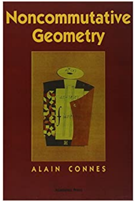
where $\text{Poiss}(\lambda) =$ *Poisson law of parameter* λ.

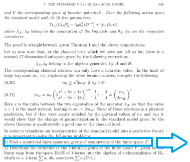
“cutoff profile”

- C*-algebra = norm closed $*$-subalgebra of $B(\mathcal{H})$ for some Hilbert $\mathcal{H}$
  = collections of "observables" in physics
- (locally compact) topological space $\Omega \leftrightarrow$ commutative C*-algebra $C_0(\Omega) \subset B(\ell_2(\Omega))$
  "quantum topological space" $\leftrightarrow$ noncommutative C*-algebra
- symmetries on topological spaces $\leftrightarrow$ topological groups
  "quantum symmetries on classical/quantum topological spaces"
  $\leftrightarrow$ topological quantum groups

- C*-algebra = norm closed $*$-subalgebra of $B(\mathcal{H})$ for some Hilbert $\mathcal{H}$
  = collections of "observables" in physics
- (locally compact) topological space $\Omega \leftrightarrow$ commutative C*-algebra $C_0(\Omega) \subset B(\ell_2(\Omega))$
  "quantum topological space" $\leftrightarrow$ noncommutative C*-algebra
- symmetries on topological spaces $\leftrightarrow$ topological groups
  "quantum symmetries on classical/quantum topological spaces"
  $\leftrightarrow$ topological quantum groups







5. THE STANDARD $U(1) \times SU(2) \times SU(3)$ MODEL 626

and $\mathcal{V}$ the corresponding space of bivector potentials. Then the following action gives the standard model with its 18 free parameters:

$$\mathrm{Tr}_\omega\left((\lambda_A\theta_A^2 + \lambda_B\theta_B^2)D^{-4}\right) + \langle\psi, D_V\psi\rangle,$$

where λ_A, λ_B belong to the commutant of the bimodule and θ_A, θ_B are the respective curvatures.

The proof is straightforward, given Theorem 1 and the above computations.

Let us now note that, at the classical level which we have not left so far, there is a natural 17-dimensional subspace given by the following restriction:

λ_A, λ_B belong to the algebra generated by $\mathcal{A}$ and $\mathcal{B}$.

The corresponding classical relation can only have a heuristic value. In the limit of large top mass m_t, i.e., neglecting the other fermion masses, one gets the following:

(6.20) $$m_t \geq \sqrt{3}m_W \text{ if } \lambda_A > 0;$$

(6.21) $$m_H = m_t\left(3\frac{x^2+8x+14}{x^2+9x+18}\right)^{1/2}, \quad \frac{1}{4}x + \frac{3}{4} = \left(\frac{m_t}{2m_W}\right)^2.$$

Here x is the ratio between the two eigenvalues of the operator λ_A, so that the value $x = 1$ is the most natural, leading to $m_t = 2m_W$. None of these relations is a physical prediction, but if they were nearly satisfied by the physical values of m_t and m_H it would show that the change of parametrization in the standard model given by the above theorem is qualitatively a good one at the classical level.

In order to transform our interpretation of the standard model into a predictive theory it is important to solve the following problems:

1) Find a nontrivial finite quantum group of symmetries of the finite space F.

2) Determine the structure of the Clifford algebra of the finite space F, given by the linear map from the space H⊕H of 1-forms into the algebra of endomorphisms of H_F which to a 1-form $\sum a_i\, db_i$ associates $\sum a_i[D, b_i]$.

"... it is important to solve the following problems:
Find a nontrivial finite quantum group of symmetries of the finite space F."

– Alain Connes

- A (classical) permutation matrix $C = [c_{ij}]_{1 \le i,j \le N} \in S_N$ is such that

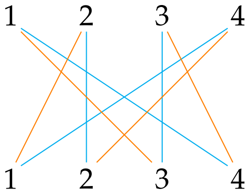
$$c_{ij} \in \{0,1\}, \quad CC^t = C^tC = I.$$

The algebra $C(S_N)$ of functions on S_N is generated by the functions $C \mapsto c_{ij}$.

- Quantum permutations (Shuzhou Wang): Consider the universal C*-algebra A generated by operators $(u_{ij})_{1 \le i,j \le N}$ s.t. for the matrix $U = [u_{ij}]_{1 \le i,j \le N}$,

$$u_{ij} = u_{ij}^* = u_{ij}^2, \quad UU^t = U^tU = I.$$

Intuitive notation: quantum permutation group $S_N^+ = (A, U)$ and $A =$"$C(S_N^+)$".



$$\begin{pmatrix} 0 & \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \\ 0 & \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} & 0 & \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \\ \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} & 0 \\ \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} & 0 & \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} & 0 \end{pmatrix}$$

Playing cards in the quantum world

- A (classical) permutation matrix $C = [c_{ij}]_{1\le i,j\le N} \in S_N$ is such that

$$c_{ij} \in \{0,1\}, \quad CC^t = C^tC = I.$$

  The algebra $C(S_N)$ of functions on S_N is generated by the functions $C \mapsto c_{ij}$.
- Quantum permutations (Shuzhou Wang): Consider the universal C*-algebra A generated by operators $(u_{ij})_{1\le i,j\le N}$ s.t. for the matrix $U = [u_{ij}]_{1\le i,j\le N}$,

$$u_{ij} = u_{ij}^* = u_{ij}^2, \quad UU^t = U^tU = I.$$

  Intuitive notation: quantum permutation group $S_N^+ = (A, U)$ and $A =$"$C(S_N^+)$".
- Interpretation in language of quantum physics: u_{ij}'s = observables measurement on a quantum state ξ $\to$ random permutation

$$\mathbb{P}(i \to j) = \langle \xi | u_{ij} | \xi \rangle$$

  Atserias,Lupini,Mancinska,Roberson,..., 19′-20′: quantum permutations much better than the classicals one when contructing strategies in non-local games on graphs.

- Analogue of group multiplications: $*$-homomorphism

$$\Delta : C(S_N^+) \to C(S_N^+) \otimes C(S_N^+), \quad u_{ij} \mapsto \sum_{k=1}^{N} u_{ik} \otimes u_{kj}.$$

- Analogue of convolutions: for two states $\varphi_1, \varphi_2 \in C(S_N^+)^*$,

$$\varphi_1 * \varphi_2 := (\varphi_1 \otimes \varphi_2) \circ \Delta.$$

- Analogue of Haar measure: $\exists$ unique state $h \in C(S_N^+)^*$ s.t. for all state $\varphi \in C(S_N^+)^*$, $\varphi * h = h * \varphi = h$, called the Haar state.
- Analogue of total variation distance: the distance in $C(S_N^+)^*$, for two states φ_1, φ_2,

$$d(\varphi_1, \varphi_2) := \frac{1}{2} \|\varphi_1 - \varphi_2\|_{C(S_N^+)^*} \Big(= \sup_{p=p^*=p^2 \in C(S_N^+)^{**}} |\varphi_1(p) - \varphi_2(p)| \Big).$$

Quantum random transpositions

Recall: classical random transpositions given by $\mu_N = \frac{N-1}{N}\mu_{\text{tran}} + \frac{1}{N}\delta_{\text{id}}$

- μ_{tran} is unif distribution on $C := \{\textit{transpositions}\}$. Note that C is a conjugacy class, so for $\mathbb{E} = |S_N|^{-1}\int \text{ad}(\sigma)d\sigma$,

$$\int_{S_N} f d\mu_{\text{tran}} = \int_{S_N} (\mathbb{E}f) d\mu_{\text{tran}} \quad \left(= (\mathbb{E}f)((12)) \right).$$

- there is a similar conditional expectation $\mathbb{E}$ from $C(S_N^+)$ onto adjoint-invariant elements. We consider analogously

$$\varphi_{\text{tran}}(f) = (\pi \circ \mathbb{E}f)((12)), \quad f \in C(S_N^+),$$

  where $\pi : C(S_N^+) \to C(S_N)$ denotes the abelianization. (Intuitively unif distribution on the quantum conjugacy class of transpositions)
- counit $\varepsilon : C(S_N^+) \to \mathbb{C}$, unique state s.t. $\varepsilon * \varphi = \varphi$, $\forall \varphi \in C(S_N^+)^*$.
- Problem: cutoff for $\varphi_N := \frac{N-1}{N}\varphi_{\text{tran}} + \frac{1}{N}\varepsilon$?

$$\varphi_N : C(S_N^+) \to \mathbb{C}, \quad \varphi_N = \frac{N-1}{N}\varphi_{\text{tran}} + \frac{1}{N}\varepsilon$$

Theorem (Freslon-Teyssier-W, PTRF 22′)

For $\epsilon > 0$, as $N \to \infty$,

$$d(\varphi_N^{*\lfloor (1-\epsilon)\frac{N\ln(N)}{2}\rfloor}, h) \to 1, \quad d(\varphi_N^{*\lfloor (1+\epsilon)\frac{N\ln(N)}{2}\rfloor}, h) \to 0.$$

Moreover we have the cutoff profile: for $c \in \mathbb{R}$, as $N \to \infty$,

$$d(\varphi_N^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor}, h)$$

$$\to d\left(D_{\sqrt{1+e^{-c}}}\left(\mathrm{Meix}^+\left(\frac{1-e^{-c}}{\sqrt{1+e^{-c}}}, \frac{-e^{-c}}{1+e^{-c}}\right)\right) * \delta_{e^{-c}}, \mathrm{Meix}^+(1,0)\right)$$

where: - $D_r(\mu)$ the r-dilation of μ (i.e. $rX \sim D_r(\mu)$ if $X \sim \mu$)

- Meix^+ denotes the free Meixner law.

Free Meixner (/Poisson/semicircular) law

Free Meixner laws are introduced by Bozejko, Bryc, Saitoh, Yoshida, as analogues of classical Meixner laws. For $a \in \mathbb{R}, b \geq 1$,

$$d\,\mathrm{Meix}^{+}(a,b)(t) = \frac{\sqrt{4(1+b)-(t-a)^2}}{2\pi(bt^2+at+1)}dt + \text{atoms}.$$

- $b = 0$: free Poisson law (i.e. Marchenko-Pastur law) for $\lambda > 1$

$$d\,\mathrm{Poiss}^{+}(\lambda,\alpha)(t) = \frac{1}{2\pi\alpha t}\sqrt{4\lambda\alpha^2-(t-\alpha(1+\lambda))^2}dt$$

- $a = b = 0$: free semicircular law $(2\pi)^{-1}\sqrt{4-t^2}dt$.

Strategy of the proof I: $\varphi_{\mathrm{tran}}^{*k}$, Case $c > 0$

$$\varphi_N : C(S_N^+) \to \mathbb{C}, \quad \varphi_N = \frac{N-1}{N}\varphi_{\mathrm{tran}} + \frac{1}{N}\varepsilon$$

- Recall that we aim to understand $d(\varphi_N^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor}, h)$.
- Consider first $d(\varphi_{\mathrm{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor}, h)$
- If $c > 0$, then $\varphi_{\mathrm{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor} \in L^1(S_N^+)$ "absolutely continuous".

$$\begin{aligned} d(\varphi_{\mathrm{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor}, h) &= \frac{1}{2}\|\varphi_{\mathrm{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor} - h\|_{L^1(S_N^+)} \\ &\leq \frac{1}{2}\|\varphi_{\mathrm{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor} - h\|_{L^2(S_N^+)} \end{aligned}$$

computable via Fourier analysis and Chebychev polynomials.

Strategy of the proof II: $\varphi_{\text{tran}}^{*k}$, Case $c < 0$

- If $c < 0$, then $\varphi_{\text{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor} \notin L^1(S_N^+)$ non absolutely continuous

STRATEGY OF THE PROOF II: $\varphi_{\text{tran}}^{*k}$, CASE $c < 0$

- If $c < 0$, then $\varphi_{\text{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor} \notin L^1(S_N^+)$ non absolutely continuous
- It suffices to consider $C(S_N^+)_{\text{central}}$ =C*-subalg generated by $\sum_i u_{ii}$.

$$d(\varphi_{\text{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor}, h) = \|(\varphi_{\text{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor} - h)|_{C(S_N^+)_{\text{central}}}\|_{C(S_N^+)^*_{\text{central}}}$$

- $C(S_N^+)_{\text{central}} \simeq C([0, N])$ $\rightsquigarrow$ a classical measure

$$\int_0^N f dm_k^{(N)} = \varphi_{\text{tran}}^{*k}(f).$$

STRATEGY OF THE PROOF II: $\varphi_{\mathrm{tran}}^{*k}$, CASE $c < 0$

- If $c < 0$, then $\varphi_{\mathrm{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor} \notin L^1(S_N^+)$ non absolutely continuous
- It suffices to consider $C(S_N^+)_{\mathrm{central}}$ =C*-subalg generated by $\sum_i u_{ii}$.

$$d(\varphi_{\mathrm{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor}, h) = \|(\varphi_{\mathrm{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor} - h)|_{C(S_N^+)_{\mathrm{central}}}\|_{C(S_N^+)^*_{\mathrm{central}}}$$

- $C(S_N^+)_{\mathrm{central}} \simeq C([0,N])$ $\rightsquigarrow$ a classical measure

$$\int_0^N f dm_k^{(N)} = \varphi_{\mathrm{tran}}^{*k}(f).$$

Proposition (Freslon-Teyssier-W)

$$m_k^{(N)} = \alpha_N(k)\delta_{\tilde{N}(k)} + \tilde{m}_k^{(N)},$$

where $\alpha_N(k) \in \mathbb{R}$ *and* $\tilde{N}(k) \notin [0,4]$, *and* $\tilde{m}_k^{(N)} \in L^2([0,4], \mathrm{Poiss}^+(1,1))$.

Strategy of the proof III: from $\varphi^{*k}_{\mathrm{tran}}$ to φ^{*k}_N

- The previous methods lead to the cutoff for $\varphi^{*k}_{\mathrm{tran}}$.
  (Contrary to the classical setting; purely quantum!)

STRATEGY OF THE PROOF III: FROM $\varphi_{\text{tran}}^{*k}$ TO φ_N^{*k}

- The previous methods lead to the cutoff for $\varphi_{\text{tran}}^{*k}$.
  (Contrary to the classical setting; purely quantum!)
- How to pass to φ_N^{*k}? $\varphi_N^{*k} \notin L^1(S_N^+)$ never absolutely continuous!

$$\varphi_N = \frac{N-1}{N}\varphi_{\text{tran}} + \frac{1}{N}\varepsilon$$

The previous Fourier analytic tools break down.

STRATEGY OF THE PROOF III: FROM $\varphi_{\text{tran}}^{*k}$ TO φ_N^{*k}

- The previous methods lead to the cutoff for $\varphi_{\text{tran}}^{*k}$.
  (Contrary to the classical setting; purely quantum!)
- How to pass to φ_N^{*k}? $\varphi_N^{*k} \notin L^1(S_N^+)$ never absolutely continuous!

$$\varphi_N = \frac{N-1}{N}\varphi_{\text{tran}} + \frac{1}{N}\varepsilon$$

  The previous Fourier analytic tools break down.
- Idea: φ_N^{*k} = randomized $\varphi_{\text{tran}}^{*k}$
  - Flip a biased coin with probability $1/N$ for heads ;
  - Tail $\rightsquigarrow$ pick φ_{tran} ; head $\rightsquigarrow$ do nothing. $X_k \sim \text{Binom}(k, \frac{N-1}{N})$
  - $\varphi_N^{*k} = \mathbb{E}(\varphi_{\text{tran}}^{*X_k})$

- The previous methods lead to the cutoff for $\varphi_{\mathrm{tran}}^{*k}$.
  (Contrary to the classical setting; purely quantum!)
- How to pass to φ_N^{*k}? $\varphi_N^{*k} \notin L^1(S_N^+)$ never absolutely continuous!

$$\varphi_N = \frac{N-1}{N}\varphi_{\mathrm{tran}} + \frac{1}{N}\varepsilon$$

  The previous Fourier analytic tools break down.
- Idea: φ_N^{*k} = randomized $\varphi_{\mathrm{tran}}^{*k}$
  - Flip a biased coin with probability $1/N$ for heads ;
  - Tail $\rightsquigarrow$ pick φ_{tran} ; head $\rightsquigarrow$ do nothing. $X_k \sim \mathrm{Binom}(k, \frac{N-1}{N})$
  - $\varphi_N^{*k} = \mathbb{E}(\varphi_{\mathrm{tran}}^{*X_k})$

Proposition (Freslon-Teyssier-W)

For $c \in \mathbb{R}$,

$$\|\varphi_N^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor} - \varphi_{\mathrm{tran}}^{*\lfloor \frac{1}{2}(N\ln(N)+cN)\rfloor}\|_{C(S_N^+)^*} \to 0, \quad \textit{as } N \to \infty.$$

Thank you very much!